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IL'IUSHIN'S POSTULATE AND RESULTING THERMODYNAMIC CONDITIONS ON ELASTO-PLASTIC COUPLING

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Abstract—Il'iushin's postulate is restated within a general thermodynamic strain space formulation of rate independent plasticity by means of plastic internal variables. This yields a general expression in terms of appropriate thermodynamic potentials. A combination of a thermodynamic condition, derived from the general development, with the results of Il'iushin's postulate, furnishes explicit conditions on elasto-plastic coupling. A specific example is presented, with the plastic work being the only plastic internal variable. Necessary and sufficient consitions on the elastic moduli and their change with plastic deformation are derived, for the thermodynamic condition to be satisfied.

1. INTRODUCTION

Il'iushin's postulate of plasticity[1] states that if the work, W, done by the external forces in an isothermal closed-cycle of deformation of an elasto-plastic material is positive, then plastic deformation takes place, and if the work is zero, only elastic deformation occurs. Stated analytically yields

$$W = \int_P \sigma_{ij}\dot{\epsilon}_{ij}\,\mathrm{d}t \geq 0 \tag{1}$$

with σ_{ij}, $\dot{\epsilon}_{ij}$ appropriate stress and strain-rate tensors and P a closed path of integration in strain space. Assuming a linear elastic behavior with the elastic moduli functions of the plastic deformation, and considering a straight line closed path in strain space, Il'iushin derived on the basis of (1) an expression for the direction of the plastic strain rate. This expression becomes the normality condition, i.e. the plastic strain rate is normal to the yield surface in stress space if the elastic moduli do not depend on the plastic deformation. This was derived earlier using Drucker's postulate[2], but Il'iushin's postulate is less restictive and has the advantage to treat simultaneously stable and unstable behavior[1, 3].

To exemplify this, consider the uniaxial stress-strain curve in Fig. 1. The material behavior is conventionally called stable along the rising part OU of the curve, and unstable along the falling part UZ. This characterization reflects the stability and instability observed during the uniaxial experiment performed with a testing machine controlling the stress, and can be analytically described and generalized by Drucker's postulate of stability[2]. If, however, the strain is controlled, which is the case in most real situations, no instability is observed during the corresponding experiment.

Il'iushin's postulate is obviously satisfied for any closed strain path starting at any point on the rising curve OU. Furthermore, it is satisfied for a similar path starting at any point on the falling curve UZ, as the path BCc indicates. The latter case cannot be obtained from Drucker's postulate, simply because no similar cycle of stress is possible. Therefore, Il'iushin's postulate applies to all paths of interest. In fact it should not be viewed as a postulate related to some kind of material stability, but merely as a generalization of the fact that the elastic modulus E is always positive and greater than the tangent modulus E' which can be positive (OU), zero (point U) or negative (UZ), precluding upon unloading a behavior shown by the path Aa'. In other words this posutulate is simply a constitutive assumption characterizing a very large class of materials behaving as shown in Fig. 1, including both "stable" and "unstable" behavior.

Proposing his postulate Il'iushin presented also a strain space formalism of plasticity[1] by simply considering the yield surface in strain space. He did not, however, give a complete strain space formulation of rate independent plasticity within thermodynamics. This is done in [4] using plastic internal variables and is summarized in the following section. A similar development was presented in [12] and to a lesser degree of similarity in [13]. A strain space formulation within a purely mechanical theory is also presented in [17], where its significance in treating stable and unstable behavior is recognized, and explained in the following.

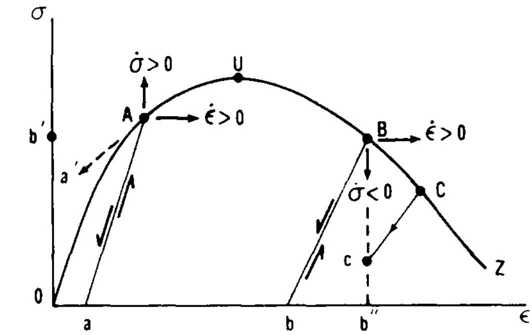

Fig. 1. Plastic loading in the uniaxial case, for stable and unstable material behavior.

The reason for a strain space formulation here is to treat simultaneously stable and unstable behavior, since Il'iushin's postulate does so. To clarify this point, recall that in the classical stress space formulation, plastic deformation occurs when the loading function $l = (\partial f/\partial \sigma_{ij})\dot{\sigma}_{ij}$ is positive, with $f = 0$ the equation of yield surface in stress space and $\dot{\sigma}_{ij}$ the stress-rate. For the case shown in Fig. 1, the yield surface at a typical point B is the projection b' of B on the σ-axis, and the loading function l degenerates to $\dot{\sigma}$. Observe that plastic deformation occurs at a stable point A where $\dot{\sigma} > 0$, but also at an unstable point B with $\dot{\sigma} < 0$. Therefore, such a definition of a loading function fails to characterize unambiguously plastic loading for unstable behavior. Observe, however, that at both points A and B the total strain rate $\dot{\epsilon}$ is always positive for plastic loading and negative for elastic unloading along Aa or Bb. This suggests a strain space formulation. The yield surface at point B in strain space is now the projection b'' or B on the ϵ-axis and the loading function is the $\dot{\epsilon}$. These notions are generalized and briefly presented in the following section.

Within this thermodynamic strain space formulation of rate independent plasticity, Il'iushin's postulate is restated. This yields general expressions in terms of appropriate thermodynamic potentials allowing for nonlinear elastic response, large deformations and any kind of plastic strain hardening by means of plastic internal variables. Further use of these expressions provides relations for the direction of the plastic strain rate in stress and strain space. The normality condition is obtained if the form of the corresponding thermodynamic potentials reflects the fact that the elastic properties are not coupled with the plastic deformation.

It is, however, observed that for many materials the elastic properties do change with plastic deformation. This is particularly true along the falling part of the stress-strain curve. Such falling part and changing elastic properties may be rare for metals but is a very common phenomenon for granular media like concrete, rocks, dense sands, overconsolidated clays, etc. The general theory of plasticity[5] provides thermodynamic conditions on the corresponding thermodynamic potentials which are derived here within the strain space-plastic internal variables formulation. Then Il'iushin's postulate is combined with thermodynamics to obtain explicit conditions on elasto-plastic coupling. A specific example is presented with the plastic work W^p being the only plastic internal variable entering an appropriate thermodynamic potential. It is further possible to derive an explicit necesasry (and sufficient for isotropic elastic behavior) condition on the elastic moduli and their change with W^p. The condition derived in this example is compared to the one obtained without use of Il'iushin's postulate but still assuming elasto-plastic coupling. It should be emphasized again that this postulate cannot be derived by thermodynamic stability considerations as shown recently in [14].

As a closing note, observe that the strain space formulation and Il'iushin's postulate are very suitable in studying coupled elasto-plastic properties; this is so because both the formulation and the postulate apply to unstable material behavior (falling stress-strain curve) and it is during such a behavior that the elasto-plastic coupling is mostly pronounced.

2. STRAIN SPACE FORMULATION OF PLASTICITY IN TERMS OF PLASTIC INTERNAL VARIABLES

A brief summary of a thermodynamic strain space formulation of rate independent plasticity in terms of plastic internal variables is presented following[4, 7, 8, 18]. We begin by writing the first and second laws of thermodynamics in the appropriate form for the field formulation of continuum mechanics

$$r-(\dot{\psi}+\dot{\theta}\eta+\dot{\eta}\theta)-Q_{K,K}+S_{KL}\dot{E}_{KL}=0 \tag{2a}$$

$$\theta\gamma=-(\dot{\psi}+\eta\dot{\theta})+S_{KL}\dot{E}_{KL}-\frac{1}{\theta}Q_K\theta_{,K}\geqslant 0 \tag{2b}$$

where E_{KL} is the Langrangian strain tensor, S_{KL} is the symmetric Piola-Kirchhoff stress tensor, θ is the temperature, Q_K is the heat flux across the surface $X_K=$ constant of the reference configuration, r, ψ, η and γ denote respectively the heat supply function, the Helmholtz (free energy) function, the entropy and the entropy production, all measured per unit reference volume, and a superposed dot designates the rate (material time derivative).

The state at each material point is described by the values of "observed" or "external" variables and of "hidden" or "internal" variables. The "external" variables here are the components of the Langrangian strain tensor E_{KL} and the temperature θ. The "internal" variables are usually taken to be scalars or components of properly invariant second rank tensors (e.g. inelastic strain, plastic work). For rate independent plasticity we call them plastic internal variables or piv for abbreviation, and denote them by q_N. Note that the index N of q_N has appropriate dimensions according to the tensorial rank of q_N (i.e. zero for scalars, double index for second rank tensors, etc.).

The constitutive relation of q_N are rate-type equations, which must be independent of the time scale used, for rate independence. Proceeding to formulate the constitutive relation for q_N, the notion of a yield surface in strain-temperature space is introduced, given analytically by

$$F(E_{KL},\theta,q_N)=0. \tag{3}$$

A set E_{KL}, θ, q_N satisfying (3) is a plastic state, and a set rendering $F<0$ is an elastic state. A loading function L is defined at a plastic state by

$$L=\frac{\partial F}{\partial E_{KL}}\dot{E}_{KL}+\frac{\partial F}{\partial\theta}\dot{\theta}. \tag{4}$$

Loading is said to occur at a plastic state when $L>0$, unloading when $L<0$ and neutral loading when $L=0$. Observe that such a definition of the loading function L unambiguously defines plastic loading for both stable and unstable behavior, according to the observations made in the introduction.

The common property characterizing the piv is that their rate $\dot{q}_N$ is different than zero at a given plastic state if $L>0$; at an elastic state or at a plastic state with $L<0$ or $L=0$, $\dot{q}_N=0$. In addition, rate independence must be satisfied for $\dot{q}_N$. The above requirements are analytically expressed by

$$\begin{aligned}&\dot{q}_N=\hat{q}_N(E_{KL},\theta,\dot{E}_{KL},\dot{\theta},q_N)H(L)\quad\text{when}\quad F=0\\&\dot{q}_N=0\quad\text{when}\quad F<0\end{aligned} \tag{5}$$

where $\hat{q}_N$ must be homogeneous of degree one in $\dot{E}_{KL}$, $\dot{\theta}$ for rate independence and $H(L)$ is the Heaviside step function defined zero at $L=0$.

We further consider the case where $\hat{q}_N$ is linear in $\dot{E}_{KL}$, $\dot{\theta}$. By standard procedures [6], based on the requirement $\hat{q}_N=0$ whenever $L=0$ for continuity of (5) throughout the strain-temperature space, follows

$$\dot{q}_N=r_N(E_{KL},\theta,q_N)\langle L\rangle\quad\text{when}\quad F=0 \tag{6}$$

with $\dot{q}_N=0$ when $F<0$, where $\langle\ \rangle$ is the Macauley bracket defining the operation $\langle L\rangle=LH(L)$. Finally, the consistency condition $\dot{F}=0$ must be satisfied for all $\dot{E}_{KL}$, $\dot{\theta}$ and $\dot{q}_N$ with $L>0$, which means that the point representing the current state remains on the changing yield surface during loading.

Consistent with our previous constitutive assumption for F, eqn (3), we assume that

$$\psi = \hat{\psi}(E_{KL}, \theta, q_N). \tag{7}$$

Consider now an arbitrary homogeneous temperature distribution $\theta_{,K} = 0$ and an unloading path from a plastic state defined by an arbitrary set of values $\dot{E}_{KL}$, $\dot{\theta}$ (it should be pointed out that right at the plastic state, $\dot{E}_{KL}$, $\dot{\theta}$ are restricted so as to render $L < 0$, but thereafter are completely independent inside the yield surface). Such an arbitrary choice of $\dot{E}_{KL}$, $\dot{\theta}$ is made compatible with the first law of thermodynamics (2a) by means of an appropriate choice of the heat supply function r. Then using (6) and (7) we finally derive by standard methods[5] from the requirement that (2b) must be always satisfied

$$S_{KL} = \frac{\partial \hat{\psi}}{\partial E_{KL}}, \quad \eta = -\frac{\partial \hat{\psi}}{\partial \theta} \tag{8}$$

and

$$-\frac{\partial \hat{\psi}}{\partial q_N} r_N \geq 0. \tag{9}$$

3. PLASTIC STRAIN AND OTHER THERMODYNAMIC POTENTIALS

The plastic strain tensor E^p_{KL} is introduced as one of the piv q_N, and it is useful to denote by q_n the remaining piv except E^p_{KL}. Thus

$$\{q_N\} = \{E^p_{KL}, q_n\} \tag{10}$$

where {*} means the set of *. According to (6)

$$\dot{E}^p_{KL} = R_{KL}(E_{IJ}, \theta, q_N)\langle L\rangle \quad \text{when} \quad F = 0 \tag{11}$$

and $\dot{E}^p_{KL} = 0$ when $F < 0$. Observe that (11) yields the plastic strain rate in terms of the total strain rate through L. Such relation can be seen in a more familiar form considering the uniaxial case for small strains. Then $\dot{\sigma} = E^t\dot{\epsilon} = E^p\dot{\epsilon}^p$, thus $\dot{\epsilon}^p = (E^t/E^p)\dot{\epsilon}$, where E^t is the tangent modulus of the stress-strain curve and E^p is the plastic modulus.

Considering the relations (6) and (11), a tensor function A_{NKL} function of E_{IJ}, θ, q_N can always be defined[4] from

$$\dot{q}_N = A_{NKL}\dot{E}^p_{KL} \tag{12}$$

which yields the system of N equations with $N \times 6$ unknowns

$$r_N = A_{NKL}R_{KL}. \tag{13}$$

If for example q_N is the E^p_{IJ} itself, then (13) yields $R_{IJ} = A_{IJKL}R_{KL}$, thus $A_{IJKL} = \delta_{IK}\delta_{JL}$ as one possible solution.

Following[5] an elastic strain tensor E^e_{KL} is introduced by

$$E^e_{KL} = E_{KL} - E^p_{KL}. \tag{14}$$

It is useful to have ψ a function of E^e_{KL}, θ and q_N. We can therefore rewrite (7) using (10) and (14) as

$$\psi = \hat{\psi}(E^e_{KL} + E^p_{KL}, \theta, E^p_{KL}, q_n) = \bar{\psi}(E^e_{KL}, \theta, q_N) \tag{15}$$

where $\bar{\psi}$ is a different function of its arguments than $\hat{\psi}$. Observe from (15) that if E^p_{KL} is not included in the q_N entering $\hat{\psi}$, is nevertheless present in the q_N entering $\bar{\psi}$, and vice-versa. From the first of (8), (10), (14) and (15), easily follows

$$\frac{\partial\hat{\psi}}{\partial E_{KL}}=\frac{\partial\bar{\psi}}{\partial E^{e}_{KL}}=S_{KL},\quad \frac{\partial\hat{\psi}}{\partial q_n}=\frac{\partial\bar{\psi}}{\partial q_n} \tag{16a}$$

$$\frac{\partial\bar{\psi}}{\partial E^{p}_{KL}}=\frac{\partial\hat{\psi}}{\partial E_{KL}}+\frac{\partial\hat{\psi}}{\partial E^{p}_{KL}}=S_{KL}+\frac{\partial\hat{\psi}}{\partial E^{p}_{KL}}. \tag{16b}$$

We further introduce two more thermodynamic potentials by appropriate Legendre transformations. A complementary free energy $\hat{g}$ per unit reference volume is defined by

$$\hat{g}=\hat{g}(S_{KL},\theta,q_N)=S_{KL}E_{KL}-\hat{\psi}(E_{KL},\theta,q_N). \tag{17}$$

In addition, a complementary elastic free energy $\bar{g}$ per unit reference volume is defined by

$$\bar{g}=\bar{g}(S_{KL},\theta,q_N)=S_{KL}E^{e}_{KL}-\bar{\psi}(E^{e}_{KL},\theta,q_N) \tag{18}$$

where $\hat{g}=\bar{g}+S_{KL}E^{p}_{KL}$.

From (16a), (17) and (18) follows

$$E_{KL}=\frac{\partial\hat{g}}{\partial S_{KL}},\quad E^{e}_{KL}=\frac{\partial\bar{g}}{\partial S_{KL}} \tag{19a}$$

$$\frac{\partial\hat{g}}{\partial q_N}=-\frac{\partial\hat{\psi}}{\partial q_N},\quad \frac{\partial\bar{g}}{\partial q_N}=-\frac{\partial\bar{\psi}}{\partial q_N}. \tag{19b}$$

Using now (10), (13), (16) and (19b), the thermodynamic condition (9) can be written in terms of the above potentials as

$$-\frac{\partial\hat{\psi}}{\partial q_N}A_{NIJ}R_{IJ}=\left(S_{IJ}-\frac{\partial\bar{\psi}}{\partial q_N}A_{NIJ}\right)R_{IJ}\geq 0 \tag{20}$$

or

$$\frac{\partial\hat{g}}{\partial q_N}A_{NIJ}R_{IJ}=\left(S_{IJ}+\frac{\partial\bar{g}}{\partial q_N}A_{NIJ}\right)R_{IJ}\geq 0. \tag{21}$$

4. IL'IUSHIN'S POSTULATE

The yield surface $F=0$ in strain space is schematically shown in Fig. 2, where the space of elastic strain E^{e}_{KL} is superposed to the space of the total strain E_{KL}. Each point in or on the $F=0$ is characterized by the same values of q_N, but has different E^{e}_{KL}. Thus, the 00′ connecting the

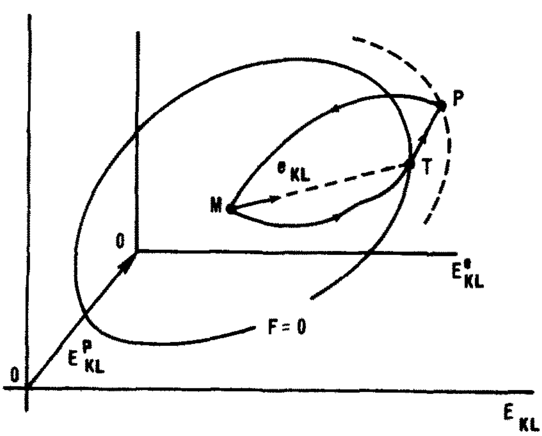

Fig. 2. Closed cycle in strain space followed by plastic loading at point T.

origins of the total and elastic strains represents the common plastic strain E^p_{KL} for each point according to (14).

Consider now an isothermal quasi-thermodynamic closed cycle of deformation at a material point, which is represented by the closed strain path *MTPM*, Fig. 2. This path has three parts. The first part *MT* carries the material point from the elastic state *M* to the plastic state *T*. The second part *TP* consists of a very small increment $\dot{E}_{KL}$ dt of the total strain, followed by plastic loading. And the third part *PM* is an elastic return to point *M* in strain space. Before proceeding into a detailed consideration of each part an important point must be brought up. During the plastic loading path *TP*, the assumption of a continuous variation of shape and position of the yield surface is made. The new position of the yield surface in the neighborhood of *T* is shown by a dashed line in Fig. 2, which passes through *P* due to the consistency condition. Observe that the yield surface has moved outward at *T*, and this is always true for both stable or unstable behavior for the strain space formulation, according to the observations made in the introduction. If, however, the stress space representation of the yield surface and Drucker's postulate of a stress cycle were considered, an inward motion of the yield surface would be obtained for unstable behavior. This apparently unimportant detail could cause problems when *M* is taken very close to *T* in the limit, as is done in the following. In that case *M* which is now the beginning and the end of a stress cycle, could be found outside the changed yield surface in stress space, and in fact it would be impossible to return to *M* within the neighborhood of *T* for unstable behavior. This was discussed in the introduction for the uniaxial case and shows the weakness of a stress space formulation together with Drucker's postulate to treat such cases. On the other hand, with Il'iushin's postulate and a strain space formulation no such problems arise and corresponding proofs of normality and convexity follow for unstable behavior as well as for a stable one.

Coming back to the closed strain path *MTPM*, the state of the material is defined as follows. State M: E^M_{KL}, q_N, θ. State T: E^T_{KL}, q_N, θ. State P: $E^T_{KL}+\dot{E}^T_{KL}\,\mathrm{d}t$, $q_N+\dot{q}_N\,\mathrm{d}t$, θ. State M' (which is point M in strain space): E^M_{KL}, $q_N+\dot{q}_N\,\mathrm{d}t$, θ, where a superscript at E_{KL} indicates the corresponding point representing the state in strain space. For an isothermal process, if follows from (7) and (8a)

$$\dot{\psi} = S_{KL}\dot{E}_{KL} + \frac{\partial\hat{\psi}}{\partial q_N}\dot{q}_N \tag{22}$$

Using (22), the work done along a path from State 1 to State 2 is given by

$$W_{12} = \int_1^2 S_{KL}\dot{E}_{KL}\,\mathrm{d}t = \psi_2 - \psi_1 - \int_1^2 \frac{\partial\hat{\psi}}{\partial q_N}\dot{q}_N\,\mathrm{d}t, \tag{23}$$

where a subscript at ψ indicates the corresponding state and the last integral from 1 to 2 is path dependent. Applying (23) to the different parts of the path *MTPM* from state M to state M' we obtain for part MT: $W_{MT}=\psi_T-\psi_M$, for part TP: $W_{TP}=\psi_P-\psi_T-(\partial\hat{\psi}_T/\partial q_N)\dot{q}_N\,\mathrm{d}t$ and for part PM: $W_{PM'}=\psi_{M'}-\psi_P$. For the part *TP* the integrand substitutes the integral in (23) within second order of approximation, due to the smallness of $\dot{E}_{KL}$ dt along *TP*. For the same reason we can write

$$\psi_{M'} - \psi_M = \frac{\partial\hat{\psi}_M}{\partial q_N}\dot{q}_N\,\mathrm{d}t \tag{24}$$

where reference was made to the previous state description at M and M'. Adding up W_{MT}, W_{TP} and W_{PM} we obtain the total work W, and with the help of (6) and (24) Il'iushin's postulate (1) yields

$$\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T)r_N \geq 0. \tag{25}$$

5. RESULTING THERMODYNAMIC CONDITIONS

A combination of the thermodynamic relation (20) or (21) with (25), yields restrictions on elasto-plastic coupling. To this end, we first use (25) to derive expressions for the direction R_{IJ} of the plastic strain rate $\dot{E}^p_{IJ}$ in strain and stress space.

(a) *Strain space*

The difference in the total and elastic strain at points T and M is given by

$$E_{KL}|_T - E_{KL}|_M = E^e_{KL}|_T - E^e_{KL}|_M = xe_{KL} \tag{26}$$

where $x \geq 0$ and e_{KL} is a unit vector in the six-dimensional strain space along the straight line MT, Fig. 2. From (15) and (26) follows

$$\hat{\psi}_M - \hat{\psi}_T = \bar{\psi}_M - \bar{\psi}_T = -\frac{\partial\hat{\psi}_T}{\partial E_{KL}}xe_{KL} + O(x^2) = -\frac{\partial\bar{\psi}_T}{\partial E^e_{KL}}xe_{KL} + O(x^2) \tag{27}$$

where $O(x^2)$ are quantities of second and higher order in x. Using (13) and (27) we can write

$$\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T)r_N = -\frac{\partial^2\hat{\psi}}{\partial E_{KL}\partial q_N}A_{NIJ}R_{IJ}xe_{KL} + O(x^2) \tag{28}$$

where the second order derivative is taken at point T. Similarly, using (10), (13), (16) and (27) we can write in terms of $\bar{\psi}$

$$\begin{aligned}\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T)r_N &= \frac{\partial}{\partial q_n}(\hat{\psi}_M - \hat{\psi}_T)r_n + \frac{\partial}{\partial E^p_{IJ}}(\hat{\psi}_M - \hat{\psi}_T)R_{IJ}\\ &= \frac{\partial}{\partial q_n}(\bar{\psi}_M - \bar{\psi}_T)r_n + \left(\frac{\partial}{\partial E^p_{IJ}} - \frac{\partial}{\partial E^e_{IJ}}\right)(\bar{\psi}_M - \bar{\psi}_T)R_{IJ}\\ &= \left(\frac{\partial^2\bar{\psi}}{\partial E^e_{KL}\partial E^e_{IJ}} - \frac{\partial^2\bar{\psi}}{\partial E^e_{KL}\partial q_N}A_{NIJ}\right)R_{IJ}xe_{KL} + O(x^2)\end{aligned} \tag{29}$$

where the second order derivatives are taken at point T. From $x > 0$, (28) and (29), Il'iushin's postulate (25) yields

$$M_{KLIJ}R_{IJ}e_{KL} + O(x) \geq 0 \tag{30}$$

with

$$M_{KLIJ} = -\frac{\partial^2\hat{\psi}}{\partial E_{KL}\partial q_N}A_{NIJ} = \frac{\partial^2\bar{\psi}}{\partial E^e_{KL}\partial E^e_{IJ}} - \frac{\partial^2\bar{\psi}}{\partial E^e_{KL}\partial q_N}A_{NIJ} \tag{31}$$

a fourth order tensor which can be arranged as a six by six square matrix. Assume now that point M approaches point T remaining always inside the yield surface, but otherwise in an arbitrary way. Then $x \to 0$ and (30) must be satisfied for all possible directions e_{KL}. This is possible only if

$$R_{IJ} = \lambda M^{-1}_{IJKL}\frac{\partial F}{\partial E_{KL}}, \quad \lambda > 0 \tag{32}$$

where $M^{-1}_{ABKL}M_{KLIJ} = \delta_{AI}\delta_{BJ}$.

(b) *Stress space*

Using (3) and (19a), the equation of the yield surface in stress space is given by

$$f(S_{KL}, \theta, q_N) = 0 \tag{33}$$

where the q_N in (33) may differ by an extra E^p_{KL} from the q_N entering (3). For example, the yield surface given in the form (33) may not include E^p_{KL} in its q_N, while using (16a) in order to convert (33) to (3) introduces E^p_{KL} as one of the q_N in (3). Considering the corresponding points M, T in stress space we can write

$$S_{KL}|_T - S_{KL}|_M = xs_{KL}, \quad x \geq 0 \tag{34}$$

where s_{KL} is a unit vector in the six-dimensional stress space along the direction MT. From $\hat{g} = \bar{g} + S_{KL}E^p_{KL}$, (19a) and (34) follows

$$\hat{g}_M - \hat{g}_T = \bar{g}_M - \bar{g}_T = -\frac{\partial \hat{g}_T}{\partial S_{KL}} x s_{KL} + O(x^2) = -\frac{\partial \bar{g}_T}{\partial S_{KL}} x s_{KL} + O(x^2). \tag{35}$$

Using (13), (19b) and (35) we obtain

$$\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T) r_N = \frac{\partial^2 \hat{g}}{\partial S_{KL}\partial q_N} A_{NIJ} R_{IJ} x s_{KL} + O(x^2) \tag{36}$$

where the second order derivative is taken at point T. Using now the first and third member of (29), (10), (13), the first of (16a), the second of (19b), (34) and (35) we obtain

$$\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T) r_N = \frac{\partial}{\partial q_N}(\bar{\psi}_M - \bar{\psi}_T) A_{NIJ} R_{IJ} + x s_{IJ} R_{IJ} = \left(\delta_{KI}\delta_{LJ} + \frac{\partial^2 \bar{g}}{\partial S_{KL}\partial q_N} A_{NIJ}\right) R_{IJ} x s_{KL} + O(x^2) \tag{37}$$

with δ_{PQ} the Kronecker delta. From $x > 0$, (36) and (37), Il'iushin's postulate (25) yields:

$$Q_{KLIJ} R_{IJ} s_{KL} + O(x) \geq 0 \tag{38}$$

with

$$Q_{KLIJ} = \frac{\partial^2 \hat{g}}{\partial S_{KL}\partial q_N} A_{NIJ} = \delta_{KI}\delta_{LJ} + \frac{\partial^2 \bar{g}}{\partial S_{KL}\partial q_N} A_{NIJ} \tag{39}$$

a fourth order tensor which can be arranged as a six by six square matrix for convenience. By an argument similar to the one which led to (32), we obtain for the direction R_{IJ} in stress space

$$R_{IJ} = \mu Q^{-1}_{IJKL} \frac{\partial f}{\partial S_{KL}}, \quad \mu > 0 \tag{40}$$

where $Q^{-1}_{ABKL} Q_{KLIJ} = \delta_{AI}\delta_{BJ}$.

Starting with a non-negative work assumption in a closed cycle of deformation, Naghdi and Trapp[15, 16] derive two local inequalities one of which is used to derive an expression for the direction of the plastic strain rate in stress space. It can be shown that (40) is identical to this expression (eqn (5.22) in [15]), if q_N are chosen to be the plastic strain E^p_{KL} and a work-hardening parameter κ where $\dot{\kappa} = H_{KL}\dot{E}^p_{KL}$. Under the same choice of q_N, the second inequality (eqn $(20)_2$ in [16]) can be obtained by multiplying (32) with $-LM_{ABIJ}\dot{E}_{AB}$ and using (31), $(8)_1$, and (4) with $\dot{\theta} = 0$. Alternatively, the second inequality in [15, 16] can be used to derive (32) when the loading function is defined by eqn (4), as in the present strain space formulation.

We can now combine the thermodynamic relations (20) and (21) with the relations (32) and (40) derived from Il'iushin's postulate, to obtain the final forms

$$-\frac{\partial \hat{\psi}}{\partial q_N} A_{NIJ} M^{-1}_{IJKL} \frac{\partial F}{\partial E_{KL}} = \left(\frac{\partial \bar{\psi}}{\partial E^e_{IJ}} - \frac{\partial \bar{\psi}}{\partial q_N} A_{NIJ}\right) M^{-1}_{IJKL} \frac{\partial F}{\partial E_{KL}} \geq 0 \tag{41}$$

or

$$\frac{\partial \hat{g}}{\partial q_N} A_{NIJ} Q^{-1}_{IJKL} \frac{\partial f}{\partial S_{KL}} = \left(S_{IJ} + \frac{\partial \bar{g}}{\partial q_N} A_{NIJ}\right) Q^{-1}_{IJKL} \frac{\partial f}{\partial S_{KL}} \geq 0. \tag{42}$$

The M_{KLIJ} and Q_{KLIJ} can be appropriately expressed in terms of $\hat{\psi}$ or $\bar{\psi}$ and $\hat{g}$ or $\bar{g}$ respectively from (31) and (39). Thus any one of the (41) and (42) will be finally expressed in terms of a single thermodynamic potential, with external variables the E_{KL} and θ only, or S_{KL} and θ only, whichever is more convenient for the purpose at hand.

The relations (41) or (42) impose restrictions on elasto-plastic coupling, provided that the structure of the corresponding thermodynamic potential assumes such coupling. In the following section, specific examples of this kind will be presented, but it is interesting to see at this point exactly the opposite, i.e. uncoupled elastic and plastic properties and their effect on the previous relations. Considering $\bar{g}$ as the corresponding potential, assume a decomposition of $\bar{g}$ of the form

$$\bar{g} = \bar{g}_1(S_{KL}, \theta) + \bar{g}_2(q_N). \tag{43}$$

It follows from (19a) and (43) that E^e_{KL} is a function of S_{KL}, θ and not of q_N, therefore no coupling exists. In addition, (39) yields $Q_{KLIJ} = \delta_{KI}\delta_{LJ}$ and (40) gives

$$R_{IJ} = \mu\frac{\partial f}{\partial S_{IJ}}, \quad \mu > 0 \tag{44}$$

which is the normality condition. This is in accordance with [1, 15, 16] but differs from [3] where the authors conclude without formal proof that normality holds true with or without coupling. The reason for the difference is that here the coupling effect is taken into account early through Q_{KLIJ} in (38) and is preserved during the limiting process where point M approaches T. Finally, the last of (42) becomes

$$\left(S_{IJ} + \frac{\partial \bar{g}_2}{\partial q_N}A_{NIJ}\right)\frac{\partial f}{\partial S_{IJ}} \geq 0. \tag{45}$$

The effect of a relation similar to (45), with a corresponding decomposition for $\bar{\psi}$, on the shape of the yield surface and the form of $\bar{\psi}_2$ has been studied in [5, 9], with E^p_{IJ} the corresponding q_N.

6. EXAMPLES OF THERMODYNAMIC CONDITIONS ON ELASTO-PLASTIC COUPLING

In the following the thermodynamic potential $\bar{g}$ will be considered a function of S_{KL}, θ and only one scalar piv denoted by q, where denoting with A_{KL} the corresponding A_{NKL}

$$\dot{q} = A_{KL}\dot{E}^p_{KL}. \tag{46}$$

Denoting with a prime the partial differentiation with respect to q, the second member of (39) yields

$$Q_{KLIJ} = \delta_{KI}\delta_{LJ} + \frac{\partial \bar{g}'}{\partial S_{KL}}A_{IJ}. \tag{47}$$

Representing the stress and strain tensors as six dimentional vectors in stress and strain space, Q_{KLIJ} can be arranged as a six by six square matrix. The quantity $B_{KLIJ} = (\partial \bar{g}'/\partial S_{KL})A_{IJ}$ considered as a square matrix B, satisfies the condition $B^2 = trB \cdot B$. Thus, the inverse of Q_{KLIJ}, according to eqn (A6) of the Appendix, is

$$Q^{-1}_{IJKL} = \delta_{IK}\delta_{JL} - \left(1 + \frac{\partial \bar{g}'}{\partial S_{QR}}A_{QR}\right)^{-1}\frac{\partial \bar{g}'}{\partial S_{IJ}}A_{KL} \tag{48}$$

with $(\partial \bar{g}'/\partial S_{QR})A_{QR} \neq -1$ for the inverse to exist.

To proceed further a specific choice of q is necessary. Here, we choose the plastic work W^p, whose rate is given by

$$\dot{W}^p = S_{KL}\dot{E}^p_{KL} \tag{49}$$

thus $A_{KL} = S_{KL}$. With W^p being the only piv entering (33), isotropic hardening is implied. Therefore, the yield surface in stress space includes the origin and we assume in addition that is convex so that $(\partial f/\partial S_{KL})S_{KL} \geq 0$. Note that the convexity cannot be proved on the basis of Il'iushin's postulate if elasto-plastic coupling exists [1], but must be assumed which is an experimental fact in most real cases. With these general assumptions and the aid of (48), the second member of the thermodynamic condition (42) yields

$$(1+\bar{g}')\left(1+\frac{\partial \bar{g}'}{\partial S_{KL}} S_{KL}\right) \geq 0 \tag{50}$$

for all S_{KL}, θ and W^p satisfying $f = 0$. If Il'iushin's postulate was not consistently employed but instead the normality condition (44) was *a priori* assumed together with the convexity of $f = 0$ and W^p was the only piv entering $\bar{g}$, the second of (21) yields

$$1+\bar{g}' \geq 0. \tag{51}$$

A comparison of (50) and (51) clearly shows the effect of Il'iushin's postulate in this case. The normality condition assumed in deriving (51), normally follows from Il'iushin's postulate if elasto-plastic coupling does not exist according to (43) and (44). But such coupling is also assumed in (51). Therefore, although it is conceivable from a general point of view to assume normality without Il'iushin's postulate, together with elasto-plastic coupling, it is believed that (50) is the result of a more consistent and rational approach than (51).

If now $\bar{g}$ is assumed to be a homogeneous function of order m in S_{KL}, so is $\bar{g}'$ and using Euler's theorem for homogeneous functions we obtain from (50)

$$(1+\bar{g}')(1+m\bar{g}') \geq 0 \tag{52}$$

thus

$$1+m\bar{g}' > 0 \tag{53a}$$

or

$$1+\bar{g}' \leq 0 \tag{53b}$$

where existence of the inverse in (48) excludes the equality sign from (53a). Since it is reasonable to assume that $\bar{g}'$ is a continuous function of S_{KL}, θ and W^p, it is impossible to find two different states of the material where (53a) is active for the one and (53b) is active for the other. The contrary would imply that the continuously varying $\bar{g}'$ assumed the values in the open interval $(-1, -1/m)$ violating the thermodynamic requirement.

In order to see a concrete application of (53), assume for $\bar{g}$ the homogeneous form of second order ($m = 2$)

$$\bar{g} = \frac{1}{2} B_{IJKL} S_{IJ} S_{KL} \tag{54}$$

where the fourth order tensor B_{IJKL} of inverse elastic moduli is a function of W^p and θ. In addition, assume that $f = 0$ depends on the deviatoric stress tensor. The Tresca and Mises yield criteria for example, are particular cases. Thus, if S^0_{IJ} is a stress state satisfying $f = 0$ and one of (53) for a given W^p and θ, it follows that $\bar{S}_{IJ} = S^0_{IJ} + p\delta_{IJ}$ must also satisfy the same conditions with p an arbitrary hydrostatic pressure. For $\bar{S}_{IJ}$, eqn (54) yields

$$\bar{g}' = \frac{1}{2} B'_{IJKL} S^0_{IJ} S^0_{KL} + p B'_{JJKL} S^0_{KL} + \frac{1}{2} p^2 B'_{JJKK} \tag{55}$$

which must satisfy the appropriate condition (53) satisfied by the first term in (55) which is the value of $\bar{g}'$ for S^0_{IJ}. It is understood that the contraction operation implied by the repeated indices JJ and KK in (55) suceeds the partial differentiation with respect to W^p. The (53a) or (53b) can be viewed as a quadratic expression in p with $\bar{g}'$ given by (55). If this quadratic expression has real roots, there will always be a range of values for the arbitrary p which violates (53). Thus, the quadratic expression must have complex roots and in that case assumes the sign of the coefficient B'_{JJKK} of p^2. Therefore, necessarily follows

$$B'_{JJKK} \gtreqless 0 \tag{56}$$

for (53a) and (53b) respectively. Observe that (56) applies to any yield condition $f = 0$ depending on the stress deviator, and is a necessary (but not sufficient) condition. In order to derive necessary and sufficient conditions, specific forms of $f = 0$ and $\bar{g}$ must be assumed.

In the following an isotropic material obeying Mises and Tresca yield conditions is assumed. The form of B_{IJKL} is

$$B_{IJKL} = -\frac{\nu}{E}\delta_{IJ}\delta_{KL} + \frac{1+\nu}{2E}(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \tag{57}$$

with the Young's modulus E and the Poisson's ratio ω functions of W_p and θ. The Mises yield criterium is given by

$$f = S_{IJ}S_{IJ} - \frac{1}{3}(S_{KK})^2 - \frac{2}{3}Y^2 = 0 \tag{58}$$

with $Y(W^p)$ the changing yield stress in a uniaxial tension/compression loading. The objective is to find conditions on E, ν and their change E', ν' with W^p, for plastic stress states satisfying (58) and one of (53). Using (54), (57) and (58), (53a) yields ($m = 2$)

$$\left[\frac{1}{3}\left(\frac{1+\nu}{E}\right)' - \left(\frac{\nu}{E}\right)'\right](S_{KK})^2 + \frac{2}{3}\left(\frac{1+\nu}{E}\right)' Y^2 + 1 > 0 \tag{59}$$

where S_{KK} is arbitrary since use of (58) was made in deriving (59).

Satisfaction of (59) for all S_{KK}, implies necessarily that both the coefficient of $(S_{KK})^2$ and the constant term in (59) must be positive, which yields respectively

$$\frac{E'}{E} + \frac{2\nu'}{1-2\nu} < 0 \tag{60}$$

and

$$\frac{E'}{E} - \frac{\nu'}{1+\nu} < \frac{3E}{2(1+\nu)Y^2} \tag{61}$$

where use of $\nu < 1/2$ was made, following from the positive definiteness of $\bar{g}$. It is interesting to note from (57) that $B'_{IJKK} = (3/E^2)[(2\nu - 1)E' - 2E\nu']$, and (56) with the > 0 sign yields (60) again. This is expected since Mises criterium belongs to the general class of yield function which provided (56) as a necessary condition. However, (60) and (61), which can be written under the compact form

$$\frac{E'}{E} < \min\left[-\frac{2\nu'}{1-2\nu},\ \frac{\nu'}{1+\nu} + \frac{3E}{2(1+\nu)Y^2}\right] \tag{62a}$$

are also sufficient conditions for (53a), since (62a) suffices for the satisfaction of (59). Similarly, eqn (53b) yields

$$\frac{E'}{E} \geq \max\left[-\frac{2\nu'}{1-2\nu},\ \frac{\nu'}{1+\nu} + \frac{3E}{(1+\nu)Y^2}\right]. \tag{62b}$$

In order to find which form of (53) or (62) is active, consider a uniaxial stress state S_{11} at yield, i.e. $S_{11} = Y(W^p)$. Then

$$\bar{g}' = -(E'/2E^2)Y^2. \tag{63}$$

In a material like concrete for example, it has been found experimentally that E decreases and ν increases with plastic deformation[10, 11]. Therefore it can be assumed

$$E' \leq 0, \quad \nu' \geq 0. \tag{64}$$

From the first of (64) is seen that $\bar{g}'$ in (63) satisfies (53a). Therefore, from the discussion following eqns (53) is concluded that the (53a) and correspondingly the (62a) are the appropriate conditions in general if $E' \le 0$. In fact, the second of (64) and (62a) render (60) alone the necessary and sufficient condition in this particular case.

Consider now the Tresca yield criterium in terms of the principal stresses

$$f = S_I - S_K - Y = 0, \qquad S_K \le S_J \le S_I \tag{65}$$

with $I \ne J \ne K \ne I$ assuming the values 1, 2, 3. Using (54), (57) and (65), (53a) yields

$$\left[\left(\frac{1+\nu}{E}\right)' - 3\left(\frac{\nu}{E}\right)'\right]\Phi + 2\left(\frac{\nu}{E}\right)'\Psi + 1 > 0 \tag{66}$$

with

$$\Phi = S_I^2 + S_J^2 + S_K^2 = 3S_K^2 + 2(x+Y)S_K + x^2 + Y^2 \tag{67a}$$

$$\Psi = \frac{1}{2}[(S_I - S_J)^2 + (S_J - S_K)^2 + (S_K - S_I)^2] = (x^2 + Y^2 - xY) \tag{67b}$$

$$0 \le x = S_J - S_K \le Y, \quad \Phi > 0, \quad \frac{3}{4}Y^2 \le \Psi \le Y^2. \tag{67c}$$

Since use of (65) was made in deriving (67), the S_K in Φ is arbitrary and because Ψ is bounded, satisfaction of (66) for all S_K yields necessarily a positive value for the coefficient of Φ. Thus, eqn (60) follows again (or (56) with the sign > 0). If now $(\nu/E)' < 0$, the combination of S_K, x which mostly enhances the negative contribution of the second term in (66) must be found in order to derive an additional sufficient and necessary condition (the case $(\nu/E)' > 0$ is thus covered). To this end observe that for any given value of x the minimum value of Φ is $(2/3)\Psi$ for $S_K = (-1/3)(x+Y)$. Therefore (66) yields

$$\frac{2}{3}\left(\frac{1+\nu}{E}\right)'\Psi + 1 > 0 \tag{68}$$

which must be satisfied for all values of Ψ, thus for its maximum Y^2 according to (67c) which yields again (61) (covering the worst possible case for $((1+\nu)/E)' < 0$). Therefore (62a) follows from (53a) for the Tresca criterium, as a necessary and sufficient thermodynamic condition on elasto-plastic coupling. Similarly (62b) follows from (53b) for the same criterium.

If Il'iushin's postulate is not employed, the active thermodynamic condition is (51) which yileds (56) with $a \ge$ sign; in particular for an isotropic material obeying Mises or Tresca yield criteria, (51) yields (62b) with a $\le$ sign and min instead of max.

Finally observe that non-existence of the inverse in (48) yields $2\bar{g}' + 1 = 0$ and from (59) follows that $\nu' = E(2\nu - 1)/2Y^2$, $E' = E^2/Y^2$.

7. CONCLUSIONS

In this presentation Il'iushin's postulate was restated within a general thermodynamic strain space formulation of rate independent plasticity be means of plastic internal variables. Both the postulate and the strain space formulation apply simultaneously to stable (rising uniaxial stress-strrain curve) and unstable (falling stress-strain curve) material behavior, as opposed to Drucker's postulate and the stress space formulation of plasticity, which fail to treat in a unified approach both behaviors.

Within this framework Il'iushin's postulate yields a general condition expressed in terms of different thermodynamic potentials. Further use of this condition yields expressions for the direction of the plastic strain rate in strain and stress space. The usual normality condition is obtained, if the form of the corresponding thermodynamic potential implies uncoupled elasto-plastic properties.

The general development yields a thermodynamic condition on the corresponding potentials which is combined with the results of Il'iushin's postulate to furnish closed form conditions on elasto-plastic coupling. This is further illustrated by a specific example where the only plastic internal variable is the plastic work W^p and the corresponding condition is compared against the result obtained without Il'iushin's postulate. Assuming further a quadratic form of the corresponding thermodynamic potential and a general form of the yield surface, it is possible to derive an explicit necessary condition on the inverse elastic moduli and their change with W^p. For isotropic elastic behavior and the Mises and Tresca yield criteria necessary and sufficient conditions on the Young's modulus and Poisson's ratio and their change with W^p are obtained for the thermodynamic condition to be satisfied. It will be of further interest to compare the present theoretical results to corresponding experimental observations on elasto-plastic coupling.

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APPENDIX

Find the inverse of the square matrix

$$Q = I + B \tag{A1}$$

when

$$B^2 = h \cdot B \tag{A2}$$

with h a scalar. We can always write

$$Q^{-1} = I + C. \tag{A3}$$

From (A1), (A3) and $QQ^{-1} = I$ follows

$$B + C + BC = 0. \tag{A4}$$

With $C = \lambda B$, (A4) yields

$$B^2 = -[(\lambda + 1)/\lambda]B. \tag{A5}$$

From (A2) and (A5) λ can be evaluated and finally

$$Q^{-1} = I - (1 + h)^{-1}B \tag{A6}$$

with $h \neq -1$. A common form of B satisfying (A2) is

$$B_{ij} = X_i Z_j \quad \text{with} \quad h = trB = X_i Z_i \tag{A7}$$

where X_i, Z_j are vectors.